Painlevé Analysis for Supersymmetric Extensions of the Sawada-Kotera Equation

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In this paper, Painlevé analysis of supersymmetric extensions of the Sawada-Kotera (SK) equation is performed. It is shown that only two simple supersymmetric extensions of the Sawada-Kotera equation pass the Painlevé test. One of them was proposed by Tian and Liu, the other one is a B-extension of the SK equation.

Key words: Supersymmetric Sawada-Kotera Equation; Supersymmetric Extension; Painlevé Analysis.

1. Introduction

The following fifth-order evolution equation,

$$u_t + u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x = 0, \qquad (1)$$

is a well-known system in soliton theory. It was proposed by Sawada and Kotera [1], also by Caudrey, Dodd, and Gibbon independently more than thirty years ago [2]. So it is referred as Sawada-Kotera (SK) equation or Caudrey-Dodd-Gibbon-Sawada-Kotera equation in literature. Now there are a lot of papers about it and thus its various properties have been established. For example, its Darboux transformation was given in [3] and [4], its bi-Hamiltonian structure was worked out by Fuchssteiner and Oevel [5], its Painlevé property was verified by Harada and Oishi [6], its Bäcklund transformation and Lax representation were provided in [7] and [8].

Soliton equations or integrable systems have many interesting extensions and one of them is the supersymmetric extension. Up to now, many equations such as Korteweg-de Vries (KdV), modified Korteweg-de Vries (mKdV), Kadomtsev-Petviashvili (KP), and nonlinear Schrödinger (NLS) equations have been embedded into their supersymmetric counterparts. The point is that these supersymmetric systems have also remarkable properties and potential applications.

Our aim is to construct supersymmetric counterparts for the SK equation. On this regard, the following equation, proposed by Carstea [9] based on the Hirota bilinear approach is known as

$$\begin{aligned}&\Phi_t + \Phi_{xxxxx}\\&+ \left[10\Phi_{xx}(\mathcal{D}\Phi) + 5\Phi(\mathcal{D}\Phi_{xx}) + 5\Phi(\mathcal{D}\Phi)^2\right]_x = 0,\end{aligned}$$

where $\Phi = \Phi(x,t,\theta)$ is a fermionic super variable depending on the super spatial variables x, θ and the usual temporal variable t. $\mathcal{D} = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}$ is the super derivative. However, there is not much known about its integrabilty apart from the two-soliton solution. Very recently, Tian and Liu [10] obtained another version for the supersymmetric SK (sSK) equation. It reads as

$$\begin{aligned}&\Phi_t + \Phi_{xxxxx} + 5\Phi_{xxx}(\mathcal{D}\Phi) + 5\Phi_{xx}(\mathcal{D}\Phi_x)\\&+ 5\Phi_x(\mathcal{D}\Phi)^2 = 0.\end{aligned} \qquad (2)$$

They showed that their system is indeed integrable. Popowicz [11] studied this system further and showed that it is an remarkable supersymmetric equation with unusal properties.

An interesting question arises: is there any other supersymmetric system which qualifies as a supersymmetric SK equation? In this paper, we will show that a general supersymmetric extension of the SK equation with free parameters is integrable only for two cases. One is the sSK equation (2) and the other is a B-extended equation [12]. We will adopt the singularity analysis or Painlevé analysis to reach above conclusion.

The article is organized as follows: In Section 2, we present the supersymmetric extensions of the SK equation containing three free parameters. In Section 3, we proceed with the Painlevé analysis for the equation obtained in Section 2. In Section 4, our conclusion is reported.

2. The Supersymmetric Extensions for the SK Equation

In this section, we are to construct a supersymmetric analogy for the SK equation. To this aim, we extend the ordinary fields $u(x,t)$ to the superfield $\Phi(x,t,\theta)$:

$$u(x,t) \to \Phi(x,t,\theta) = \theta u(x,t) + \xi(x,t),$$

where θ is a Grassmannian variable: $\theta^2 = 0$, and $\xi(x,t)$ is a fermionic field. Then we proceed with a direct extension of the SK equation, multiplying each term by θ and rewriting the results of superfields:

$$\begin{aligned}
&u_t \to \Phi_t,\\
&u^2 u_x \to k_3 \Phi(\mathcal{D}\Phi)(\mathcal{D}\Phi_x) + (5-k_3)\Phi_x(\mathcal{D}\Phi)^2,\\
&u_x u_{xx} \to k_2 \Phi_x(\mathcal{D}\Phi_{xx}) + (5-k_2)\Phi_{xx}(\mathcal{D}\Phi_x),\\
&u u_{xxx} \to k_1 \Phi(\mathcal{D}\Phi_{xxx}) + (5-k_1)\Phi_{xxx}(\mathcal{D}\Phi),\\
&u_{xxxxx} \to \Phi_{xxxxx},
\end{aligned}$$

where k_1, k_2, k_3 are the free parameters. We notice that the nonlinear terms do not have the unique extensions in the term of superfields. Therefore, this direct extension supplies us with a supersymmetric version of the SK equation containing three free parameters:

$$\begin{aligned}
&\Phi_t + \Phi_{xxxxx} + k_1 \Phi(\mathcal{D}\Phi_{xxx}) + (5-k_1)\Phi_{xxx}(\mathcal{D}\Phi)\\
&+ k_2 \Phi_x(\mathcal{D}\Phi_{xx}) + (5-k_2)\Phi_{xx}(\mathcal{D}\Phi_x)\\
&+ (5-k_3)\Phi_x(\mathcal{D}\Phi)^2 + k_3 \Phi(\mathcal{D}\Phi)(\mathcal{D}\Phi_x) = 0.
\end{aligned} \tag{3}$$

In components, it reads as:

$$\begin{aligned}
&u_t + u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x\\
&-k_1 \xi_{xxxx} + (5-k_1-k_2)\xi_x \xi_{xxx} - k_3 u \xi \xi_{xx}\\
&-k_3 u_x \xi \xi_x = 0,
\end{aligned} \tag{4a}$$

$$\begin{aligned}
&\xi_t + \xi_{xxxxx} + k_1 u_{xxx}\xi + k_2 u_{xx}\xi_x\\
&+ k_3 \xi u u_x + (5-k_1) u \xi_{xxx} + (5-k_2) u_x \xi_{xx}\\
&+ (5-k_3) u^2 \xi_x = 0.
\end{aligned} \tag{4b}$$

It is clear that this system does reduce to the SK equation when the fermionic variable is absent. This equation is our general sSK equation.

3. Painlevé Analysis

In this section, we perform the Painlevé analysis for the system (4a–4b). To this end, we consider an expansion of the component fields about a movable singular manifold $\phi(x,t) = 0$:

$$\begin{aligned}
u(x,t) &= \sum_{j=0}^{\infty} u_j(x,t)\phi(x,t)^{j-\alpha},\\
\xi(x,t) &= \sum_{j=0}^{\infty} \xi_j(x,t)\phi(x,t)^{j-\beta},
\end{aligned} \tag{5}$$

where $\xi_j(x,t)$ are fermionic fields. In order to simplify the calculation, we use the Kruskal's ansatz [13, 14]:

$$\begin{aligned}
&u_j(x,t) = u_j(t), \quad \xi_j(x,t) = \xi_j(t),\\
&\phi(x,t) = x + f(t),
\end{aligned} \tag{6}$$

where $f(t)$ is an arbitrary function. Besides, we command $u_0 \neq 0$ and $\xi_0 \neq 0$.

3.1. Leading Order Analysis

In this step, we start with determining the possible values of α and β in the expression (5). By substituting $u \approx u_0\phi^{-\alpha}, \xi \approx \xi_0\phi^{-\beta}$ into (4a–4b), we find $\alpha = 2$ and β is an arbitrary integer.

3.2. Resonance Structure Analysis

It is known that a 'resonance' occurs at j when the coefficient $u_j(t)$ of the term $\phi^{j-\alpha}$ or the coefficient $\xi_j(t)$ of the term $\phi^{j-\beta}$ in the expression (5) is arbitrary. Substituting (5), (6), and $\alpha = 2$ into the system (4a–4b) leaves us with the following recursion formula:

$$\begin{aligned}
0 &= u_{j-5,t} + (j-6)u_{j-4}f'\\
&+ 5(j-6)\sum_{n=0}^{j}(n-2)(n-3)u_n u_{j-n}\\
&-k_3 \sum_{n=0}^{j}\sum_{m=0}^{n+2\beta-3}(m-\beta)(j+m-n-\beta-3)\xi_{n+2\beta-3-m}\\
&\cdot \xi_m u_{j-n} + (5-k_1-k_2)\sum_{n=0}^{j+2\beta-3}(j+\beta-3-n)(n-\beta)\\
&\cdot (n-\beta-1)(n-\beta-2)\xi_{j+2\beta-3-n}\xi_n\\
&+ 5\sum_{n=0}^{j}\left(\sum_{m=0}^{n} u_n u_{n-m}\right)(j-n-2)u_{j-n} + (j-2)\\
&\cdot (j-3)(j-4)(j-5)(j-6)u_j - k_1 \sum_{n=0}^{j+2\beta-3}(n-\beta)\\
&\cdot (n-\beta-1)(n-\beta-2)(n-\beta-3)\xi_{j+2\beta-3-n}\xi_n,
\end{aligned} \tag{7}$$

$$
\begin{aligned}
0 &= \xi_{j-5,t} + (j-\beta-4)\xi_{j-4}f' \\
&+ k_3 \sum_{n=0}^{j}\left(\sum_{m=0}^{j-n}(m-2)u_m u_{j-n-m}\right)\xi_n \\
&+ (5-k_3)\sum_{n=0}^{j}\left(\sum_{m=0}^{j-n}u_n u_{j-n-m}\right)(n-\beta)\xi_n \\
&+ k_2\sum_{n=0}^{j}(n-\beta)(j-n-2)(j-n-3)u_{j-n}\xi_n \\
&+ k_1\sum_{n=0}^{j}(j-n-2)(j-n-3)(j-n-4)u_{j-n}\xi_n \\
&+ (5-k_2)\sum_{n=0}^{j}(n-\beta)(n-\beta-1)(j-n-2)u_{j-n}\xi_n \\
&+ (5-k_1)\sum_{n=0}^{j}(n-\beta)(n-\beta-1)(n-\beta-2)u_{j-n}\xi_n \\
&+ (j-\beta)(j-\beta-1)(j-\beta-2) \\
&\quad \cdot (j-\beta-3)(j-\beta-4)\xi_j.
\end{aligned}
\tag{8}
$$

Solving these equations we must remememember that $u_j(t)$ and $\xi_j(t)$ are even and odd elements of the Grassman algebra, respectively, i. e.

$$
u_j = d_j + U_j, \quad d_j \in \mathbb{C}, \quad U_j \text{ is nilpotent}, \quad \xi_j^2 = 0.
$$

For $j=0$, we get

$$
0 = u_0^3 + 18u_0^2 + 72u_0 + \sigma_1 u_0 - \sigma_2 - \sigma_3, \tag{9}
$$

$$
\begin{aligned}
0 = \xi_0 \Big\{ &\left[(5-k_3)\beta + 2k_3\right]u_0^2 \\
&+ \beta(\beta+1)(\beta+2)(\beta+3)(\beta+4) \\
&+ \left[(5-k_1)\beta(\beta+1)(\beta+2)\right. \\
&\left.+ 2(5-k_2)\beta(\beta+1) + 6k_2\beta + 24k_1\right]u_0 \Big\}.
\end{aligned}
\tag{10}
$$

For $\beta \le 1$, we have

$$
\sigma_1 = \sigma_2 = \sigma_3 = 0, \quad u_0 = -12, \quad \text{or} \quad u_0 = -6.
$$

For $\beta \ge 2$, we obtain

$$
\begin{aligned}
\sigma_1 &= \frac{k_3}{10}\sum_{m=0}^{2\beta-3}(m-\beta)(m-\beta-3)\xi_{2\beta-3-m}\xi_m, \\
\sigma_2 &= -\frac{k_1}{10}\sum_{n=0}^{2\beta-3}(n-\beta)(n-\beta-2)(n-\beta-3) \\
&\qquad \cdot (n-\beta-4)\xi_{2\beta-3-n}\xi_n, \\
\sigma_3 &= \frac{1}{10}(5-k_1-k_2)\sum_{n=0}^{2\beta-3}(\beta-3-n)(n-\beta) \\
&\qquad \cdot (n-\beta-1)(n-\beta-2)\xi_{2\beta-3-n}\xi_n,
\end{aligned}
$$

$$
\sigma_i^\beta = 0, \quad i = 1,2,3,
$$

and

$$
u_0 = d_0 + U_0 = -6 + \frac{1}{2}(1+\sqrt{3}\mathrm{i})(X+Y)
$$

or

$$
u_0 = d_0 + U_0 = -12 + \frac{1}{2}(1-\sqrt{3}\mathrm{i})(X+Y),
$$

where

$$
\begin{aligned}
X &= -\frac{\sqrt{3}}{3}(\sqrt{3}+\mathrm{i})KL, \\
Y &= -\frac{\sqrt{3}}{3}(\sqrt{3}-\mathrm{i})KLV + \frac{\sqrt{3}(\sqrt{3}+\mathrm{i})}{36}\sigma_1 V, \\
V &= 1 + \frac{1}{3}KL + (\frac{1}{3}KL)^2 + (\frac{1}{3}KL)^3 + \ldots, \\
K &= \\
&\frac{1}{5184\mathrm{i}\sqrt{3}}(2592\mathrm{i}\sqrt{3}PH - 648\sigma_1 - 108\sigma_2 - 108\sigma_3), \\
L &= 1 + \frac{1}{3}K + \frac{5}{9}K^2 + \frac{10}{81}K^3 + \ldots, \\
H &= 1 + \frac{1}{4}P + \frac{1}{4}P + \frac{1}{8}P^3 + \ldots, \\
P &= -\frac{1}{559872}\Big(46656\sigma_1 + 1620\sigma_1^2 + 12\sigma_1^3 \\
&\qquad + 972\sigma_1\sigma_2 + 972\sigma_3\sigma_1 + 81\sigma_2^2 \\
&\qquad + 162\sigma_3\sigma_2 + 81\sigma_3^2\Big)
\end{aligned}
$$

with $\mathrm{i}^2 = -1$. (10) implies that ξ_0 is an arbitrary odd Grassmania function of t.

Now we determine resonances and recursion relations for u_j and ξ_j. We can rewrite (7) and (8) in the form:

$$
\begin{aligned}
F_j &= u_j P(j) + X_j \xi_0 \xi_{j+2\beta-3}, \\
G_j &= \xi_j Q(j) + Y_j \xi_0 u_j,
\end{aligned}
$$

where

$$
\begin{aligned}
&P(j) = (j-2)(j-3)(j-4)(j-5)(j-6) \\
&+ 5(j-6)(j^2-5j+12)u_0 + 5(j-6)u_0^2 \\
&- k_3\sum_{m=0}^{2\beta-3}(m-\beta)(j+m-\beta-3)\xi_{2\beta-3-m}\xi_m, \\
&Q(j) = \Big[(5-k_3)(j-\beta) - 2k_3\Big]u_0^2 \\
&+ \Big[(5-k_1)(j-\beta)(j-\beta-1)(j-\beta-2) \\
&-2(5-k_2)(j-\beta)(j-\beta-1) + 6k_2(j-\beta) - 24k_1\Big]u_0 \\
&+ (j-\beta)(j-\beta-1)(j-\beta-2)(j-\beta-3)(j-\beta-4),
\end{aligned}
$$

$$X_j = (5-k_1-k_2)\beta(j+\beta-3)\Big[(\beta+1)(\beta+2) -(j+\beta-4)(j+\beta-5)\Big] + k_1\Big[(j+\beta-3)(j+\beta-4) \cdot (j+\beta-5)(j+\beta-6) - \beta(\beta+1)(\beta+2)(\beta+3)\Big],$$

$$Y_j = (5-k_2)\beta(\beta+1)(j-2) + k_3(j-4)u_0 -2\beta(5-k_3)u_0 + k_1(j-2)(j-3)(j-4) -(5-k_1)\beta(\beta+1)(\beta+2) - k_2\beta(j-2)(j-3),$$

$$F_j = -5\sum_{n=1}^{j-1}\Big(\sum_{m=1}^{n} u_m u_{n-m}\Big)(j-n-2)u_{j-n} - u_{j-5,t} -(j-6)u_{j-4}f' - 5(j-6)\sum_{n=1}^{j-1}(n-2)(n-3)u_n u_{j-n} + k_3\sum_{n=1}^{j-1}\Big(\sum_{m=1}^{n+2\beta-3}(m-\beta)(j+m-n-\beta-3) \cdot \xi_{n+2\beta-3-m}\xi_m\Big)u_{j-n} + k_1\sum_{n=1}^{j+2\beta-4}(n-\beta)(n-\beta-1) \cdot (n-\beta-2)(n-\beta-3)\xi_{j+2\beta-3-n}\xi_n - (5-k_1-k_2) \cdot \sum_{n=1}^{j+2\beta-4}(j+\beta-3-n)(n-\beta)(n-\beta-1) \cdot (n-\beta-2)\xi_{j+2\beta-3-n}\xi_n,$$

$$G_j = -k_2\sum_{n=1}^{j-1}(n-\beta)(j-n-2)(j-n-3)u_{j-n}\xi_n -(5-k_1)\sum_{n=1}^{j-1}(n-\beta)(n-\beta-1)(n-\beta-2)u_{j-n}\xi_n -(5-k_2)\sum_{n=1}^{j-1}(n-\beta)(n-\beta-1)(j-n-2)u_{j-n}\xi_n -k_3\sum_{n=1}^{j-1}\Big(\sum_{m=1}^{j-n}(m-2)u_m u_{j-n-m}\Big)\xi_n -(5-k_3)\sum_{n=1}^{j-1}\Big(\sum_{m=1}^{j-n}u_m u_{j-n-m}\Big)(n-\beta)\xi_n - \xi_{j-5,t} -(j-\beta-4)\xi_{j-4}f' -k_1\sum_{n=1}^{j-1}(j-n-2)(j-n-3)(j-n-4)u_{j-n}\xi_n.$$

Thus, the resonances of the system (4a–4b) are the roots of the equations $P(j)=0$ and $Q(j)=0$, respectively. We will discuss the resonances and the corresponding values of the parameters k_1,k_2,k_3 for the case $\beta \le 1$ and the case $\beta \ge 2$ separately.

For $\beta \le 1$, $P(j)=0$, we have

$$(j-2)(j-3)(j-4)(j-5)(j-6) +5(j-6)(j^2-5j+12)u_0 + 5(j-6)u_0^2 = 0. \quad (11)$$

In this case, we have $u_0=-12$ or $u_0=-6$. In the case $u_0=-12$, the set of the roots of (11) is $\{-2,-1,5,6,12\}$, which contains two negative integers, so (4a–4b) could not possess the generalized Painlevé property. Therefore, we take $u_0=-6$.

Substituting $u_0=-6$ into the equations $P(j)=0$ and $Q(j)=0$, respectively, gives

$$P'(j) = (j+1)(j-2)(j-3)(j-6)(j-10), \quad (12)$$

$$Q'(j) = (j+2-\beta)\Big[\beta^4 + (12-4j)\beta^3 + \big(6j^2-36j+29+6k_1\big)\beta^2 + \big(-4j^3+36j^2-58j -12k_1 j + 30k_1 + 12k_2 + 42\big)\beta + \big(j^4-12j^3+29j^2 + 6k_1 j^2 - 12k_2 j + 42j - 30k_1 j - 36k_3 - 72k_1\big)\Big]. \quad (13)$$

Then the resonances of recursion relations (7) and (8) are roots of $P'(j)=0$ (i. e. $j=-1,2,3,6,10$) and roots j_1,j_2,j_3,j_4,j_5 of $Q'(j)=0$, where the resonance $j=-1$ corresponds to the arbitrary function ϕ defining in the singular manifold.

For $\beta=1$, substituting $\beta=1$ and $u_0=-6$ into (10), we have

$$108k_1 + 12k_2 - 36k_3 = 0, \quad (14)$$

and (13) gives

$$Q'(j) = j(j+1)\Big[j^3 - 16j^2 + (6k_1+71)j -(42k_1+12k_2+56)\Big]. \quad (15)$$

Suppose that three roots of the cubic equation in the square brackets in (15) are j_1,j_2,j_3. Then we obtain three equations from (15):

$$j_1+j_2+j_3 = 16, \quad (16)$$

$$j_1j_2 + j_1j_3 + j_2j_3 = 6k_1 + 71, \quad (17)$$

$$j_1j_2j_3 = 42k_1 + 12k_2 + 56. \quad (18)$$

Combining the last two equations with (14), we have the values of k_1,k_2,k_3 in terms of the three undetermined roots j_1,j_2,j_3:

$$k_1 = \frac{1}{6}(j_1j_2 + j_1j_3 + j_2j_3 - 71), \quad (19)$$

$$k_2 = \frac{1}{12}[j_1j_2j_3 - 7(j_1j_2+j_1j_3+j_2j_3) + 441], \quad (20)$$

Table 1. $j_4 = -1, j_5 = 0$.

Case	$(j_1, j_2, j_3, j_4, j_5)$	(k_1, k_2, k_3)
(a.1)	$(1,2,13,-1,0)$	$(-5,15,-10)$
(a.2)	$(1,3,12,-1,0)$	$(-\frac{10}{3},10,-\frac{20}{3})$
(a.3)	$(1,4,11,-1,0)$	$(-2,6,-4)$
(a.4)	$(1,5,10,-1,0)$	$(-1,3,-2)$
(a.5)	$(1,6,9,-1,0)$	$(-\frac{1}{3},1,-\frac{2}{3})$
(a.6)	$(1,7,8,-1,0)$	$(0,0,0)$
(a.7)	$(2,3,11,-1,0)$	$(-\frac{5}{3},\frac{20}{3},-\frac{25}{9})$
(a.8)	$(2,4,10,-1,0)$	$(-\frac{1}{2},\frac{15}{4},-\frac{1}{4})$
(a.9)	$(2,5,9,-1,0)$	$(\frac{1}{3},\frac{5}{3},\frac{14}{9})$
(a.10)	$(2,6,8,-1,0)$	$(\frac{5}{6},\frac{5}{12},\frac{95}{36})$
(a.11)	$(3,4,9,-1,0)$	$(\frac{2}{3},2,\frac{8}{3})$
(a.12)	$(3,5,8,-1,0)$	$(\frac{4}{3},\frac{2}{3},\frac{38}{9})$
(a.13)	$(3,6,7,-1,0)$	$(\frac{5}{3},0,5)$
(a.14)	$(4,5,7,-1,0)$	$(2,0,6)$

Table 2. $j_4 = -2, j_5 = 0$.

Case	$(j_1, j_2, j_3, j_4, j_5)$	(k_1, k_2, k_3)
(b.1)	$(1,2,9,-2,0)$	$(0,5,0)$
(b.2)	$(1,3,8,-2,0)$	$(1,3,2)$
(b.3)	$(1,4,7,-2,0)$	$(\frac{5}{3},\frac{5}{3},\frac{10}{3})$
(b.4)	$(1,5,6,-2,0)$	$(2,1,4)$
(b.5)	$(2,3,7,-2,0)$	$(2,2,4)$
(b.6)	$(2,4,6,-2,0)$	$(\frac{5}{2},\frac{5}{4},5)$
(b.7)	$(3,4,5,-2,0)$	$(3,1,6)$

$$k_3 = \frac{1}{36}[j_1 j_2 j_3 + 2(j_1 j_2 + j_1 j_3 + j_2 j_3) - 198]. \quad (21)$$

Now we need to find the values of j_1, j_2, j_3. Since they are positive distinct integers, we can list up all the possible combinations of them satisfying the constraint (16). In this way, we obtain Table 1 with $j_4 = -1$ and $j_5 = 0$.

For $\beta = 0$, substituting $\beta = 0$ and $u_0 = -6$ into (10), we get

$$144k_1 - 72k_3 = 0,$$

and (13) in this case yields

$$Q'(j) = (j+2)j\big[j^3 - 12j^2 + (29+6k_1)j + (-12k_2 - 30k_1 + 42)\big].$$

Similarly, we get Table 2 with $j_4 = -2$ and $j_5 = 0$.

For $\beta \geq 2$, as a result of $u_0 = d_0 + U_0$, we can write $P(j) = 0$ and $Q(j) = 0$ in the following form:

$$P''(j) = (j-2)(j-3)(j-4)(j-5)(j-6) + 5(j-6)(j^2 - 5j + 12)d_0 + 5(j-6)d_0^2 + p,$$

$$Q''(j) = \Big[(5-k_3)(j-\beta) - 2k_3\Big]d_0^2 + \Big[(5-k_1)(j-\beta)(j-\beta-1)(j-\beta-2) - 2(5-k_2)(j-\beta)(j-\beta-1) + 6k_2(j-\beta) - 24k_1\Big]d_0 + (j-\beta)(j-\beta-1)(j-\beta-2)(j-\beta-3)(j-\beta-4) + q,$$

where

$$p = 10(j-6)U_0^2 + 5(j-6)(j^2 - 5j + 12)U_0 - k_3 \sum_{m=0}^{2\beta-3} (m-\beta)(j+m-\beta-3)\xi_{2\beta-3-m}\xi_m,$$

$$q = 2\Big[(5-k_3)(j-\beta) - 2k_3\Big]U_0^2 + \Big[(5-k_1)(j-\beta)(j-\beta-1)(j-\beta-2) - 2(5-k_2)(j-\beta)(j-\beta-1) + 6k_2(j-\beta) - 24k_1\Big]U_0.$$

The contribution to the resonances is the non-fermioic part of $P''(j) = 0$ (also with $Q''(j) = 0$), namely

$$(j-2)(j-3)(j-4)(j-5)(j-6) + 5(j-6)(j^2 - 5j + 12)d_0 + 5(j-6)d_0^2.$$

Since we have known $d_0 = -12$ or $d_0 = -6$, similarly to (11), we take $d_0 = -6$. So the resonances of the recursion relations (7) and (8) are also roots of $P'(j) = 0$ and roots of $Q'(j) = 0$.

For $\beta = 2$, substituting $\beta = 2$ and $u_0 = -6$ into (10), we get

$$0 \equiv 0.$$

Substituting $\beta = 2$ and $u_0 = -6$ into (13), we get

$$j\big[j^4 - 20j^3 + (6k_1 + 125)j^2 - (54k_1 + 250 + 12k_2)j + (144 + 24k_2 + 156k_1 - 36k_3)\big].$$

Similarly to Table 1, we get Table 3 with $j_5 = 0$.

3.3. Resonance Criteria Analysis

As the last step, we have to check the resonance criteria for $\beta = 0, 1, 2$. We take advantage of the symbolic software Maple 11. The final result is that only the three cases (a.6), (b.1), and (c.15), see Tables 1–3, have satisfied all criteria.

Therefore, the system (4a–4b) has the Painlevé property only for

case (i): $k_1 = k_2 = k_3 = 0$ and

case (ii): $k_1 = k_3 = 0, k_2 = 5$.

Table 3. $j_4 = 0, j_5 = 0$.

Case	$(j_1, j_2, j_3, j_4, j_5)$	(k_1, k_2, k_3)	Case	$(j_1, j_2, j_3, j_4, j_5)$	(k_1, k_2, k_3)
(c.1)	$(1,2,3,14,0)$	$(-5,15,-10)$	(c.2)	$(1,2,4,13,0)$	$\left(-\frac{10}{3},10,-\frac{20}{3}\right)$
(c.3)	$(1,2,5,12,0)$	$(-2,6,-4)$	(c.4)	$(1,2,6,11,0)$	$(-1,3,-2)$
(c.5)	$(1,2,7,10,0)$	$\left(-\frac{1}{3},1,-\frac{2}{3}\right)$	(c.6)	$(1,2,8,9,0)$	$(0,0,0)$
(c.7)	$(1,3,4,12,0)$	$\left(-\frac{5}{3},\frac{20}{3},-\frac{25}{9}\right)$	(c.8)	$(1,3,5,11,0)$	$\left(-\frac{1}{2},\frac{15}{4},-\frac{1}{4}\right)$
(c.9)	$(1,3,6,10,0)$	$\left(\frac{1}{3},\frac{5}{3},\frac{14}{9}\right)$	(c.10)	$(1,3,7,9,0)$	$\left(\frac{5}{6},\frac{5}{12},\frac{95}{36}\right)$
(c.11)	$(1,4,5,10,0)$	$\left(\frac{2}{3},2,\frac{8}{3}\right)$	(c.12)	$(1,4,6,9,0)$	$\left(\frac{4}{3},\frac{2}{3},\frac{38}{9}\right)$
(c.13)	$(1,4,7,8,0)$	$\left(\frac{5}{3},0,5\right)$	(c.14)	$(1,5,6,8,0)$	$(2,0,6)$
(c.15)	$(2,3,4,11,0)$	$(0,5,0)$	(c.16)	$(2,3,5,10,0)$	$(1,3,2)$
(c.17)	$(2,3,6,9,0)$	$\left(\frac{5}{3},\frac{5}{3},\frac{10}{3}\right)$	(c.18)	$(2,3,7,8,0)$	$(2,1,4)$
(c.19)	$(2,4,5,9,0)$	$(2,2,4)$	(c.20)	$(2,4,6,8,0)$	$\left(\frac{5}{2},\frac{5}{4},5\right)$
(c.21)	$(2,5,6,7,0)$	$(3,1,6)$	(c.22)	$(3,4,5,8,0)$	$(3,2,5)$
(c.23)	$(3,4,6,7,0)$	$\left(\frac{10}{3},\frac{5}{3},\frac{50}{9}\right)$			

For case (i), we get the sSK equation

$$\begin{aligned}&\Phi_t + \Phi_{xxxxx} + 5\Phi_{xxx}(\mathcal{D}\Phi) + 5\Phi_{xx}(\mathcal{D}\Phi_x)\\&+ 5\Phi_x(\mathcal{D}\Phi)^2 = 0.\end{aligned} \tag{22}$$

For case (ii), we obtain the sSK equation

$$\begin{aligned}&\Phi_t + \Phi_{xxxxx} + 5\Phi_{xxx}(\mathcal{D}\Phi) + 5\Phi_x(\mathcal{D}\Phi_{xx})\\&+ 5\Phi_x(\mathcal{D}\Phi)^2 = 0.\end{aligned} \tag{23}$$

Equation (22) is identical to (2), which was proposed by Kai and Liu while (23) is the B-extension of the SK equation.

To complete the Painlevé analysis of the two cases, we must check the resonance of their other branches.

For case (i), from (10) and (13), the resonances are given by

1. $\beta = -6, \quad j_1 = -8, \quad j_2 = -7,$
   $j_3 = -6, \quad j_4 = 0, \quad j_5 = 1;$
2. $\beta = -7, \quad j_1 = -9, \quad j_2 = -8,$
   $j_3 = -7, \quad j_4 = -1, \quad j_5 = 0;$

and for case (ii), from (10) and (13), the resonances are

1. $\beta = -1, \quad j_1 = -3, \quad j_2 = -1,$
   $j_3 = 0, \quad j_4 = 1, \quad j_5 = 8;$
2. $\beta = -2, \quad j_1 = -4, \quad j_2 = -2,$
   $j_3 = -1, \quad j_4 = 0, \quad j_5 = 7;$
3. $\beta = -9, \quad j_1 = -11, \quad j_2 = -9,$
   $j_3 = -8, \quad j_4 = -7, \quad j_5 = 0.$

One can convince oneself that the compatibility conditions for these branches are satisfied.

However, there are still other possibilities that must be considered, namely those with $\beta \geq 3$. Here we only prove that the compatibility condition is contradictory for $\beta = 3$. Substituting $u_0 = -6$ and $\beta = 3$ into (10), we get

$$k_3 = 15 + 6k_1 + k_2. \tag{24}$$

From (7), $j = -2\beta + 4$, one obtains the condition

$$\begin{aligned}&2\xi_0\xi_1(\beta+1)(k_2\beta^2 - 5\beta^2 + 3\beta^2 k_1\\&+ 3k_1\beta - k_2\beta + 5\beta + k_3u_0) = 0,\end{aligned} \tag{25}$$

and then it is easy to see that

$$\xi_0\xi_1 = 0$$

which is in contradiction with the arbitrariness of ξ_0 and ξ_1, because of $j = 1$ and $j = 0$ are two roots of $Q'(j) = 0$ with $\beta = 3$.

So, the result of the Painlevé analysis is that the sSK equations possesses Painlevé property if and only if $k_1 = k_2 = k_3 = 0$ and $k_1 = k_3 = 0, k_2 = 5$. That is we only obtain two integrable supersymmetric extensions of the SK equations, and one is the sSK equation (2) and the other is the B-extension of the SK equation.

4. Discussions

We have shown that the sSK equation containing three parameters possesses the Painlevé property for $k_1 = k_2 = k_3 = 0$ and $k_1 = k_3 = 0, k_2 = 5$, which correspond to the sSK equation (2) and a B-extension of the SK equation, respectively. So no new integrable sSK equation could arise by the means of the Painlevé analysis and the one proposed by Castera does not appear in our study in particular. As a further problem, it would be interesting to study systematically the

Painlevé property of the supersymmetric extensions of the fifth-order KdV type equations in the form

$$u_{xxxxx}+a_1uu_{xxx}+a_2u_xu_{xx}+a_3u^2u_x+a_4u_t=0,$$

where a_1,a_2,a_3 are parameters. We may return to this problem later and hope some new supersymmetric equations of fifth order will appear.

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